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TOPOLOGICAL DIRECT SUM DECOMPOSITIONS OF BANACH SPACES

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ABSTRACT

Let Y and Z be two closed subspaces of a Banach space X such that $Y \neq \{0\}$ and $Y + Z = X$. Then, if Z is weakly countably determined, there exists a continuous projection T in X such that $\|T\| = 1$, $T(X) \supset Y$, $T^{-1}(0) \subset Z$ and $\operatorname{dens} T(X) = \operatorname{dens} Y$. It follows that every Banach space X is the topological direct sum of two subspaces X_1 and X_2 such that X_1 is reflexive and $\operatorname{dens} X_2^{**} = \operatorname{dens} X^{**}/X$.

The vector spaces used here are defined on the field K of the real or complex numbers. If X is a Banach space, let X^* denote its dual and X^{**} its bidual. $\|\cdot\|$ denotes the norm in X. As usual, we shall identify X with a subspace of X^{**}. If A is a subset of X, A_s stands for A endowed with the topology induced by the weak one in X, $\tilde{A}$ for the weak-star closure of A in X^{**}, $A^{\perp}$ for the subspace of X^* orthogonal to A, and A° for the polar subset of A in X^*. If M is a subset of X^*, M_σ denotes M endowed with the topology induced by the weak-star one defined on X^*, and $M_{\perp}$ the subspace of X orthogonal to M. If x belongs to X and u to X^*, $\langle x, u \rangle$ stands for $u(x)$. Let X_μ^* be the vector space X^* endowed with the topology of the uniform convergence on the absolutely convex weakly compact subsets of X; it is clear that the weak-star closure of a subspace of X^* coincides with its closure in X_μ^*.

When K denotes the field of the real numbers, H stands for the field of the rational ones; if, instead, K is the field of the complex numbers, H denotes the field of all the numbers $a + bi$, a and b rational.

If A is a set, $|A|$ is its cardinal number. The density character of a topological space E is the cardinal number λ of the first ordinal number such that there exists a dense subset A of E with $|A| = \lambda$. We shall write $\operatorname{dens} E := \lambda$.

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According to Vasak [6], let us say that a Banach space is weakly countably determined if there exists a sequence (M_n) of bounded closed absolutely convex neighbourhoods of the origin in X such that, given an arbitrary $x \in X$, there exists a decreasing subsequence (M_{n_j}) of (M_n) such that

$$x \in \bigcap_{j=1}^{\infty} \tilde{M}_{n_j} \subset X.$$

Every weakly K-analytic Banach space, in particular every weakly compactly generated Banach space, is weakly countably determined.

The following proposition will be used in the proof of Theorem 1.

Proposition. *Let (M_n) be a decreasing sequence of bounded closed and absolutely convex subsets of a Banach space X. If*

$$\bigcap_{n=1}^{\infty} \tilde{M}_n \subset X,$$

the set $\bigcup_{n=1}^{\infty} M_n^{\circ}$ is a neighbourhood of the origin in X_{μ}^{*}.

Proof. Given

$$u \in X^*, \qquad u \notin \bigcup_{n=1}^{\infty} M_n^{\circ},$$

let us choose, for every positive integer n,

$$x_n \in M_n, \qquad |\langle x_n, u\rangle| > 1.$$

The sequence (x_n) is bounded in X, hence it has a weak-star cluster point x_0 in X^{**}. Obviously,

$$x_0 \in \bigcap_{n=1}^{\infty} \tilde{M}_n \subset X,$$

and $|\langle x_0, u\rangle| \geq 1$. It follows that

$$\left\{ v \in X^* : |\langle x, v\rangle| < 1, \, x \in \bigcap_{n=1}^{\infty} \tilde{M}_n \right\} \subset \bigcup_{n=1}^{\infty} M_n^{\circ}.$$

Since $\bigcup_{n=1}^{\infty} \tilde{M}_n$ is an absolutely convex weakly compact subset of X, the conclusion follows. q.e.d.

Theorem 1. *Let Y and Z be two closed subspaces of a Banach space X such that $Y \neq \{0\}$ and $Y + Z = X$. Then, if Z is weakly countably determined, there ex-*

ists a continuous projection T in X such that $\|T\| = 1$, $T(X) \supset Y$, $T^{-1}(0) \subset Z$ and dens $T(X) =$ dens Y.

PROOF. Let α be the density character of Y. Let A_0 be a dense subset of Y such that $|A_0| = \alpha$. We can choose in $Z^{\perp}$ a weak-star dense subset B_0 such that $|B_0| \leq \alpha$.

Let us choose in Z a sequence (M_n) of closed absolutely convex and bounded neighbourhoods of the origin in such a way that, given an arbitrary vector z of Z, there exists a decreasing subsequence (M_{n_j}) of (M_n) such that

$$z \in \bigcap_{j=1}^{\infty} \tilde{M}_{n_j} \subset Z.$$

Let us denote by $|\cdot|_n$ the Minkowski functional of M_n and M_n°, M_n° being the polar set of M_n in X^*, $n = 1,2,\ldots$.

Given positive integers r and s and arbitrary vectors $x \in X$, $u \in X^*$, let us choose

$$v(x) \in X^*, \qquad z(u,r,s) \in M_r$$

such that

$$\|v(x)\| = |z(u,r,s)|_r = 1,$$

$$\langle x, v(x)\rangle = \|x\|, \qquad |\langle z(u,r,s), u\rangle| \geq \frac{s-1}{s}\,|u|_r.$$

We shall define inductively two sequences of subsets (A_n) and (B_n), $A_n \subset X$ and $B_n \subset X^*$, $n = 0,1,2,\ldots$. A_0 and B_0 have been selected before. Suppose now that we have defined $A_1,\ldots,A_n$, $B_1,\ldots,B_n$, in such a way that $|A_i| = \alpha$ and $|B_i| \leq \alpha$, $i = 0,1,2,\ldots,n$. Let C_n and D_n be the linear hull in H of the sets A_n and B_n, respectively. Let us define

$$A_{n+1} = C_n \cup \{z(u,r,s) : u \in D_n, \qquad r,s = 1,2,\ldots\},$$

$$B_{n+1} = D_n \cup \{v(x) : x \in C_n\}.$$

Denote by E and F the closures of $\cup_{n=0}^{\infty} A_n$ and $\cup_{n=0}^{\infty} B_n$ in X and X_{σ}^*, respectively. Obviously, E and F are vector spaces.

Choose x in E, z in $F_{\perp}$ and $\epsilon > 0$. Select a positive integer m and a vector t in A_m such that $\|x - t\| < \epsilon$. Then

$$\|x\| \leq \|x - t\| + \|t\| < \epsilon + \|t\| = \epsilon + \langle t, v(t)\rangle = \epsilon + \langle t + z, v(t)\rangle$$

$$\leq \epsilon + |\langle t - x, v(t)\rangle| + |\langle z + x, v(t)\rangle| \leq \epsilon + \|t - x\| + \|z + x\| \leq \|z + x\| + 2\epsilon,$$

and, therefore,

$$\|x\| \leq \|x+z\|.$$

Hence $E \cap F_\perp = \{0\}$, and the projection T of $E + F_\perp$ onto E along $F_\perp$ has norm equal to one.

Let us suppose now that there exists v in $E^\perp \cap F$, $v \neq 0$. Choose a vector x_0 in X such that $\langle x_0, v\rangle = 3$. It follows that

$$x_0 = y_0 + z_0, \qquad y_0 \in Y \subset E, \quad z_0 \in Z,$$

hence $\langle z_0, v\rangle = 3$. We can select a decreasing subsequence (M_{n_j}) of (M_n) such that

$$z_0 \in M := \bigcap_{j=1}^{\infty} \tilde{M}_{n_j} \subset Z.$$

Obviously,

$$(1) \qquad (v + M^\circ) \cap M^\circ = \varnothing.$$

According to Proposition 1, $\bigcup_{j=1}^{\infty} \overset{\circ}{M}_{n_j}$ is a neighbourhood of the origin in X_μ^*, hence there exists a member M_r of the sequences (M_{n_j}) and a vector $w \in X^*$ such that

$$(2) \qquad w \in (v + \overset{\circ}{M}_r) \cap \left(\bigcup_{n=0}^{\infty} B_n\right).$$

It follows from (1) and (2) that

$$|w|_r > 1, \qquad |w - v|_r \leq 1.$$

Let s be a positive integer such that

$$\frac{s-1}{s}\,|w|_r > 1.$$

Then, $z(w,r,s)$ belongs to E and, therefore,

$$1 \geq |w-v|_r \geq |\langle z(w,r,s), w - v\rangle| = |\langle z(w,r,s), w\rangle| \geq \frac{s-1}{s}\,|w|_r > 1,$$

a contradiction. It follows that $E + F_\perp = X$. Finally,

$$T(X) = E \supset Y, \quad T^{-1}(0) = F_\perp \subset Z, \quad \|T\| = 1, \quad \operatorname{dens} T(X) = \operatorname{dens} Y. \quad \text{q.e.d.}$$

REMARK. The method used above is suggested by a simple proof, given in [3] (a proof which holds also in the context of Fréchet spaces), of a result due to D. Amir and J. Lindenstrauss on the projective resolution of the identity operator in a weakly compactly generated Banach space [1]. If, in Theorem 1, Y is a finite-dimensional Banach space, it follows that X is weakly countably determined. In this situation, the former reasoning can be adapted, using a standard transfinite-induction procedure, to get straightforwardly a result of Vasak on projective resolution of the identity in weakly countably determined Banach spaces [6].

The following proposition will be used in the proof of Theorem 2.

PROPOSITION 2. *Let U, V and W be three closed subspaces of a Banach space E. If V is reflexive and complemented in E, with U as its topological complement, then*

$$\operatorname{dens} W/(V \cap W) \leq \operatorname{dens} U + \operatorname{dens} E/W.$$

PROOF. Let φ be the canonical mapping from E onto $E/(V \cap W)$. This last space is isomorphic to the topological direct sum of $\varphi(U)$ and $\varphi(V)$, and, since $\varphi(U)$ is isomorphic to U, it follows that

$$\operatorname{dens} E/(V \cap W) = \operatorname{dens} \varphi(U) + \operatorname{dens} \varphi(V) = \operatorname{dens} U + \operatorname{dens} \varphi(V). \tag{3}$$

Let ϕ be the canonical mapping from $E/(V \cap W)$ onto $(E/(V \cap W))/\varphi(W)$. Since

$$\varphi(V) \cap \varphi(W) = \{0\},$$

the restriction Λ of ϕ to the closed unit ball B of $\varphi(V)$ is injective and continuous, so that, by the reflexivity of V, B is weak compact and therefore the mapping

$$\Lambda : B_s \to \phi(B)_s$$

is a homeomorphism, hence

$$\operatorname{dens} \varphi(V) \leq \operatorname{dens}(E/(V \cap W))/\varphi(W).$$

Obviously, $(E/(V \cap W))/\varphi(W)$ is isomorphic to E/W, hence

$$\operatorname{dens} \varphi(V) \leq \operatorname{dens} E/W.$$

Finally, from (3) and (4) we get

$$\operatorname{dens} W/(V \cap W) \leq \operatorname{dens} E/(V \cap W) \leq \operatorname{dens} U + \operatorname{dens} E/W. \qquad \text{q.e.d}$$

THEOREM 2. *Let Y be a closed subspace of a Banach space X. If X/Y is reflexive, then there exists a complemented subspace Z of X such that*

$$Z \supset Y, \qquad \operatorname{dens} Z^{**} = \operatorname{dens} Y^{**}.$$

PROOF. Obviously, we can suppose $Y \neq \{0\}$. $Y^{\perp}$ is a reflexive subspace of X^* and $X^*/Y^{\perp}$ is isomorphic to Y^*, hence, denoting by φ the canonical mapping from X^* onto $X^*/Y^{\perp}$ and by M the open unit ball of this quotient space, we can find a dense subset $\{u_i : i \in I\}$ of M such that

$$|I| = \operatorname{dens} Y^*.$$

Given $i \in I$, let us choose v_i in the unit ball of X^* such that $\varphi(v_i) = u_i$. Denoting by F the closed linear hull of $\{v_i : i \in I\}$ in X^*, φ applies the unit ball of F onto a dense subset of M. The Closed Graph Theorem gives then $\varphi(F) = X^*/Y$. It follows that $Y^{\perp} + F = X^*$.

Using Theorem 1 we can get a closed subspace G of X^* which has a complement H such that

$$\operatorname{dens} G = \operatorname{dens} F, \qquad H \subset Y^{\perp}.$$

Since $H_{\perp}$ contains Y, it follows that $X/H_{\perp}$ is reflexive, hence X^{**}/X, Y^{**}/Y and $(H_{\perp})^{**}/H_{\perp}$ are all isomorphic. On the other hand, H is weak-star closed in X^*, hence $H^{\perp}$ coincides with the weak-star closure of $H_{\perp}$ in X^{**}. Therefore G^*_{σ} is weak-star isomorphic to $(H^{\perp})_{\sigma}$ and $G^{\perp}$ is reflexive. It follows that

$$\operatorname{dens}(H^{\perp})_{\sigma} \leq \operatorname{dens} F,$$

hence

$$\operatorname{dens} H_{\perp} \leq \operatorname{dens} Y^{**}.$$

Therefore

$$\begin{aligned} \operatorname{dens} Y^{**} \leq \operatorname{dens}(H_{\perp})^{**} &= \operatorname{dens} H_{\perp} + \operatorname{dens}(H_{\perp})^{**}/H_{\perp} \\ &\leq \operatorname{dens} Y^{**} + \operatorname{dens} Y^{**}/Y = \operatorname{dens} Y^{**}, \end{aligned}$$

and, since $H^{\perp}$ is isomorphic to $(H_{\perp})^{**}$, it follows that

$$\operatorname{dens} H^{\perp} = \operatorname{dens} Y^{**}.$$

Using now Proposition 2 in the case $U = H^{\perp}$, $V = G^{\perp}$, $W = X$ and $E = X^{**}$ we get $V \cap W = G_{\perp}$, hence

$$\operatorname{dens} X/G_{\perp} \leq \operatorname{dens} H^{\perp} + \operatorname{dens} X^{**}/X = \operatorname{dens} Y^{**} + \operatorname{dens} Y^{**}/Y = \operatorname{dens} Y^{**}.$$

Let us proceed now as at the beginning of the proof: we can find a closed subspace L of X such that

$$L + G^{\perp} = X, \qquad \operatorname{dens} L = \operatorname{dens} X/G_{\perp} \leq \operatorname{dens} Y^{**}.$$

Let T be the closure of $L + Y$ in X. Then

$$\operatorname{dens} T \leq \operatorname{dens} Y^{**}.$$

Using Theorem 1 we get a complemented subspace Z of X such that $Z \supset T$, and $\operatorname{dens} Z = \operatorname{dens} T$. This Z satisfies the required conditions. q.e.d.

To prove the following theorem we shall need the following result [4]: (a) *Let Y be a closed subspace of a Banach space X. Then $X + \tilde{Y}$ is a Banach space.*

THEOREM 3. *Every Banach space X is the topological direct sum of two subspaces X_1 and X_2 such that X_1 is reflexive and*

$$\operatorname{dens} X_2^{**} = \operatorname{dens} X^{**}/X.$$

PROOF. Obviously, it is enough to prove the result in the case

$$\operatorname{dens} X > \operatorname{dens} X^{**}/X.$$

Let α be the density character of X^{**}/X. Let $\mathfrak{F}$ be the family of all the closed subspaces F of X such that $\operatorname{dens} F = \alpha$. Put

$$G := U\{\tilde{F} : F \in \mathfrak{F}\}.$$

G is a closed subspace of X^{**}. We can find a subset $\{x_i : i \in I\}$ of G such that the linear hull of $X \cup \{x_i : i \in I\}$ is dense in G and $|I| \leq \alpha$.

Given $i \in I$ we can find F_i in $\mathfrak{F}$ such that $x_i \in \tilde{F}_i$. Let Y be the closed linear hull of $\cup\{F_i : i \in I\}$ in X. Then $\operatorname{dens} Y = \alpha$ and $X + \tilde{Y}$ is dense in G. Using now the result (a), it follows that $X + \tilde{Y} = G$.

Let φ be the canonical mapping from X onto X/Y. We shall identify, as usual, $(X/Y)^*$ with $Y^{\perp}$. Let (x_n) be a sequence in the open unit ball of X/Y. We can find y_n in the unit ball of X in such a way that $\varphi(y_n) = x_n$, $n = 1,2,\ldots$. The sequence (y_n) has a weak-star cluster point y in X^{**}. Obviously, y belongs to G. Let $y = z + t$, $z \in X$, $t \in \tilde{Y}$. We shall prove that $\varphi(z)$ is a weak cluster point of (x_n) in X/Y: take ϵ an arbitrary positive number and an arbitrary vector u in $Y^{\perp}$. There exists a positive integer n_0 such that

$$|\langle y - y_n, u\rangle| < \epsilon,$$

for infinitely many n.

Taking into account that $\tilde{Y}$ coincides with $(Y^{\perp})^{\perp}$, it follows that

$$|\langle \varphi(z) - x_n, u\rangle| = |\langle z - y_n, u\rangle| = |\langle y - y_n, u\rangle| < \epsilon$$

for infinitely many n, hence X/Y is reflexive. Theorem 2 gives now a subspace X_2 of X with a topological complement X_1 such that

$$X_2 \supset Y, \qquad \operatorname{dens} X_2^{**} = \operatorname{dens} Y^{**}.$$

Obviously, X_1 is reflexive and

$$\operatorname{dens} X_2^{**} = \operatorname{dens} Y^{**} = \operatorname{dens} Y + \operatorname{dens} Y^{**}/Y = \alpha + \operatorname{dens} X^{**}/X = \operatorname{dens} X^{**}/X.$$

q.e.d.

COROLLARY [5]. *Let X be a Banach space such that X^{**}/X is separable. Then X is the topological direct sum of two subspaces X_1 and X_2 such that X_1 is reflexive and X_2^{**} is separable.*

In [2], A. Sersouri provides an explicit decomposition of the type we consider above in the case of a quasi-reflexive space which is isometric to its bidual.

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